Deterministic Graph Spectral Sparsification

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1. Motivation

Many techniques in data analysis need to compute the eigenvalues of the data matrix. For example, in principal component analysis (PCA) or spectral clustering, to name a few.

It is well know that computation of eigenvalues of general matrices is computationally expensive, and therefore, may authors use techniques of numerical approximation. Futhermore, computations are more efficient whenever the matrices are sparse.

2. Spectral Sparsification: (Spielman and Teng- 2010)

Spectral sparsification is a technique for constructing sparse matrices (matrices with many zeros). Given a square and symetric dense matrix M (a matrix with few zeros), spectral sparsification approximates M by a sparse matrix M' with no loss of the spectral properties of M.

3. Main concepts

Let $G = (V, E)$ be an undirected and weighted graph and let $A = (A_{ab}) \in \mathbb{R}^{n\times n}$ be its adjacency matrix. We define the *Laplacian* matrix $L_G = (L_{ab})$ of G as

$$L_{ab} = \begin{cases} -A_{ab} & if \;\; a \neq b \\ \sum_b A_{ab} & if \;\; a = b. \end{cases}$$

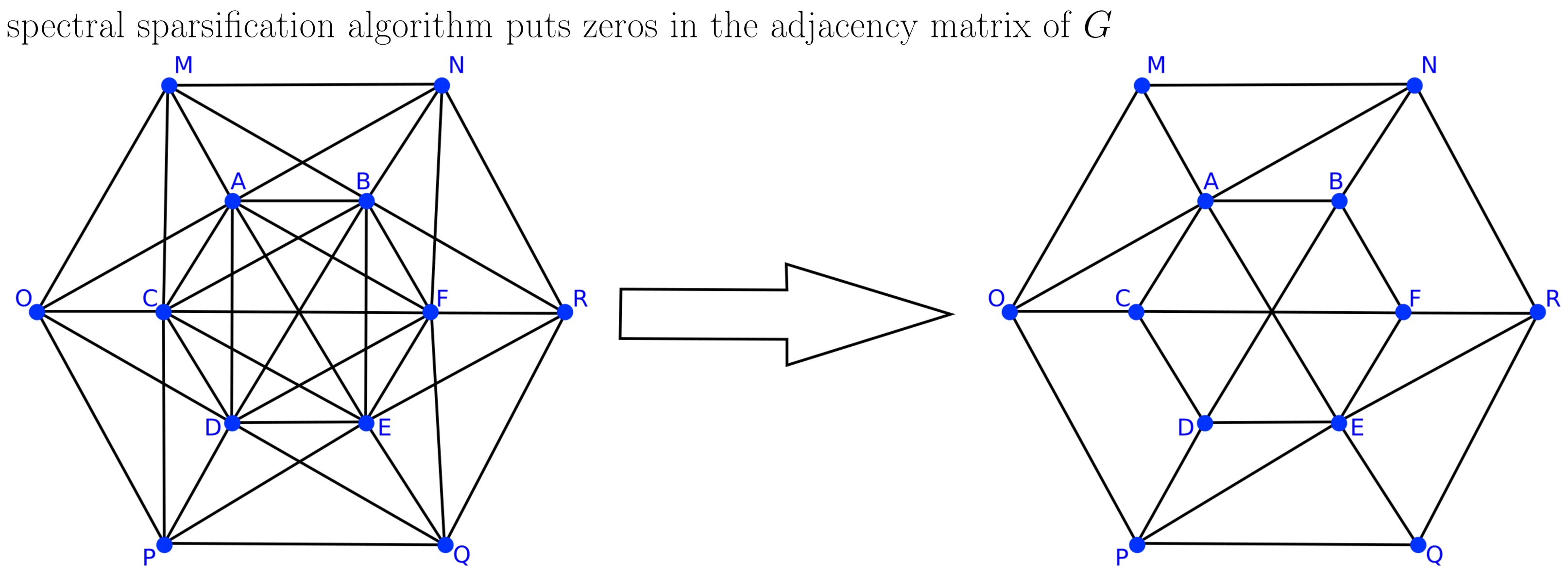
The quadratic Laplacian form of L_G is defined as

$$x^T L_G x = \sum_{(u,v)\in E} (x(u) - x(v))^2,$$

where $x \in \mathbb{R}^V = \{x : V \to \mathbb{R}\}$. We say a graph G is a σ-spectral sparsifier of G if for all $x \in \mathbb{R}^V$ it holds that

$$\frac{1}{\sigma} x^T L_{\widetilde{G}} x \leq x^T L_G x \leq \sigma x^T L_{\widetilde{G}} x.$$

A spectral sparsification algorithm puts zeros in the adjacency matrix of G

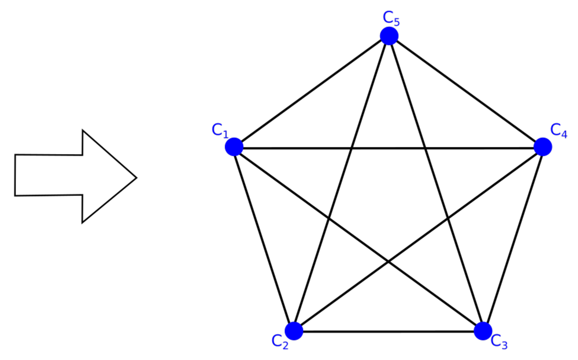


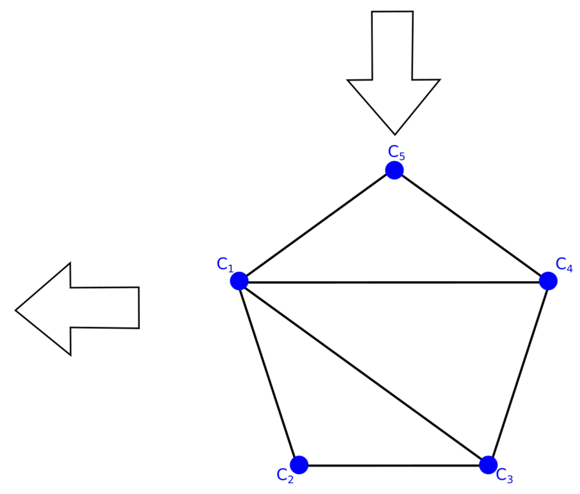
Let $\lambda_1, \lambda_2, ..., \lambda_n$ and $\phi_1, \phi_2, ..., \phi_n$ be the eigenvalues of G and $\widetilde{G}$ respectivly. If $\widetilde{G}$ is a σ-spectral sparsifier of G, then

$$\frac{1}{\sigma}\phi_i \leq \lambda_i \leq \sigma\phi_i$$

4. Example of the Spectral Sparsification Process

$$M = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{45} & c_{55} \end{pmatrix}$$





$$M' = \begin{pmatrix} c'_{11} & c'_{12} & c'_{13} & c'_{14} & c'_{15} \\ c'_{21} & c'_{22} & c'_{23} & 0 & 0 \\ c'_{31} & c'_{32} & c'_{33} & c'_{34} & 0 \\ c'_{41} & 0 & c'_{43} & c'_{44} & c'_{45} \\ c'_{51} & 0 & 0 & c'_{45} & c'_{55} \end{pmatrix}$$

5. Algorithms that have been proposed

Algorithm	Sparsifier Size	Runing time	Type
SS-11	$O(nlog(n/\varepsilon))$	$O(mlog^c m)$	Probabilistic
Zou-12	$O(n/\varepsilon^2)$	$O(mn^2/\varepsilon + n^4/\varepsilon^4)$	Deterministic
BSS-14	$O(n/\varepsilon^2)$	$O(mn^3/\varepsilon^2)$	Probabilistic
AZLO-15	$O(\sqrt{q}n/\varepsilon^2)$	$O(n^{2+\varepsilon})$	Probabilistic

✓(SS-11) Spielman - Srivastava
✓(Zou-12) Anastasios Zouzias
✓(BSS-14) Batson - Spielman - Srivastava
✓(AZLO-15) Allen - Zhenyu Liao - Lorenzo Orecchia

The algorithm with best running time is probabilistic, AZLO-15, while the best deterministic algorithm is ZOU-12. We will focus on the Zouzias algorithm given that our intention is to propose a deterministic algorithm.

6. Objective

In this work we propose to find a new deterministic method for finding spectral sparsifiers. To that end, we will study several restrictions to the adjacency matrix in order to decrease the number of deleted edges and improve the execution time of Zouzia's– algorithm[1]. This method could be used as a preprocessing step before any other application that requires the computation of eigenvalues, for example, clustering, PCA, etc.

References

[1] A. Zouzias. A matrix hyperbolic cosine algorithm and applications. In *Proceedings of the 39th International Colloquium on Automata, Languages, and Programming (ICALP)*, 2012.

[2] D. A. Spielman, and S.H.Teng. *Spectral sparsification of graphs.* SIAM Journal on Computing, 40(4) : 981-1025, 2011.

[3] D. Peña, Análisis de datos Multivariantes, S.A. McGraw-Hil/ INTERAMERICANA DE ESPAÑA. ISBN: 8448136101, 2002.

[4] Y. T. Lee, and H. Sun. Constructing linear-sized spectral sparsification in almost-linear time. In *Proceedings of the 56th Annual Symposium on Foundations of Computer Science (FOCS)*, IEEE, 2015.

[5] V. Simoncini and D. B. Szyld, Recent computational developments in Krylov subspace methods for linear systems, Numer. Linear Algebra Appl., 14:1-59, 2007.

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