Journal of Geometric Mechanics

https://www.aimspress.com/journal/jgm

Journal of Geometric Mechanics, 15(1): 287–318
DOI: 10.3934/jgm.2023012
Received: 06 July 2022
Revised: 20 September 2022
Accepted: 20 September 2023
Published: 21 February 2023

Research article

A Herglotz-based integrator for nonholonomic mechanical systems

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Abstract: We propose a numerical scheme for the time-integration of nonholonomic mechanical systems, both conservative and nonconservative. The scheme is obtained by simultaneously discretizing the constraint equations and the Herglotz variational principle. We validate the method using numerical simulations and contrast them against the results of standard methods from the literature.

Keywords: nonholonomic systems; nonconservative systems; Herglotz principle; contact integrator
Mathematics Subject Classification: Primary: 37M15, 65D30; Secondary: 70G45.

A. Discrete equations of the contact integrator with constraints for the falling disk

The following equations enumerated from (A.1) to (A.5) correspond to the resulting equations of the integrator based on the discretization of the Herglotz variational principle with constraints that we propose in this work, applied to the problem of the falling disk described in Section **??**. Here we use a second-order discretization, according to Remark **??**.

$$
\begin{aligned}
&R\cos(\phi_j)\left[\frac{F_j^X}{2} - \frac{1}{\frac{\alpha h}{2}+1}\left(\frac{F_j^X}{2} - \frac{m(X_{j-1}-X_j)}{h^2}\right)\left(\frac{\alpha h}{2}-1\right) + \frac{m(X_j - X_{j+1})}{h^2}\right] \\
&+ R\sin(\phi_j)\left[\frac{F_j^Y}{2} - \frac{1}{\frac{\alpha h}{2}+1}\left(\frac{F_j^Y}{2} - \frac{m(Y_{j-1}-Y_j)}{h^2}\right)\left(\frac{\alpha h}{2}-1\right) + \frac{m(Y_j - Y_{j+1})}{h^2}\right] \\
&- \frac{1}{\frac{\alpha h}{2}+1}\left(\frac{\alpha h}{2}-1\right)\left[\frac{F_j^\psi}{2} - \frac{1}{h} I_A\left(\frac{\psi_{j-1}-\psi_j}{h} - \sin\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\frac{\phi_{j-1}-\phi_j}{h}\right)\right] \\
&+ \frac{1}{h} I_A\left(\frac{\psi_j - \psi_{j+1}}{h} - \sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j - \phi_{j+1}}{h}\right) + \frac{F_j^\psi}{2} = 0.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
&R\cos\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\frac{\psi_j-\psi_{j+1}}{h}-\frac{X_j-X_{j+1}}{h}\\
&-R\cos\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j-\phi_{j+1}}{h}\\
&-R\cos\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\sin\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\frac{\theta_j-\theta_{j+1}}{h}=0.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&R\sin\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\frac{\psi_j-\psi_{j+1}}{h}-\frac{Y_j-Y_{j+1}}{h}\\
&-R\sin\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j-\phi_{j+1}}{h}\\
&+R\cos\left(\frac{\phi_j+\phi_{j+1}}{2}\right)\cos\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\theta_j-\theta_{j+1}}{h}=0.
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&\frac{F_j^\theta}{m}+\frac{I_T}{m}\left(\frac{2(\theta_j-\theta_{j+1})}{h^2}-\frac{\sin(\theta_j+\theta_{j+1})}{2}\frac{(\phi_j-\phi_{j+1})^2}{h^2}\right)+Rg\sin(\theta_j)\\
&+R^2\left(\sin^2\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{2(\theta_j-\theta_{j+1})}{h^2}+\frac{\sin(\theta_j+\theta_{j+1})}{2}\frac{(\theta_j-\theta_{j+1})^2}{h^2}\right)\\
&+\frac{\alpha h-2}{\alpha h+2}\left\{\frac{I_T}{m}\left[\frac{2(\theta_{j-1}-\theta_j)}{h^2}+\frac{\sin(\theta_{j-1}+\theta_j)}{2}\frac{(\phi_{j-1}-\phi_j)^2}{h^2}\right]-\frac{F_j^\theta}{m}-Rg\sin(\theta_j)\right.\\
&+R^2\left[\sin^2\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\frac{2(\theta_{j-1}-\theta_j)}{h^2}-\frac{\sin(\theta_{j-1}+\theta_j)}{2}\frac{(\theta_{j-1}-\theta_j)^2}{h^2}\right]\\
&\left.+\frac{I_A(\phi_{j-1}-\phi_j)}{m}\cos\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\left[\frac{\psi_{j-1}-\psi_j}{h^2}-\sin\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\frac{\phi_{j-1}-\phi_j}{h^2}\right]\right\}\\
&+2R\cos(\phi_j)\cos(\theta_j)\left[\frac{F_j^Y}{2m}-\frac{\alpha h-2}{\alpha h+2}\left(\frac{F_j^Y}{2m}-\frac{Y_{j-1}-Y_j}{h^2}\right)+\frac{Y_j-Y_{j+1}}{h^2}\right]\\
&-2R\cos(\theta_j)\sin(\phi_j)\left[\frac{F_j^X}{2m}-\frac{\alpha h-2}{\alpha h+2}\left(\frac{F_j^X}{2m}-\frac{X_{j-1}-X_j}{h^2}\right)+\frac{X_j-X_{j+1}}{h^2}\right]\\
&-\frac{I_A(\phi_j-\phi_{j+1})}{m}\cos\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\left[\frac{\psi_j-\psi_{j+1}}{h^2}-\sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j-\phi_{j+1}}{h^2}\right]=0.
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
&\frac{F_j^\phi}{2}+I_T\cos^2\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j-\phi_{j+1}}{h^2}\\
&-\frac{\alpha h-2}{\alpha h+2}\left\{\frac{F_j^\phi}{2}-I_T\cos^2\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\frac{\phi_{j-1}-\phi_j}{h^2}\right.\\
&\left.+I_A\sin\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\left[\frac{\psi_{j-1}-\psi_j}{h^2}-\sin\left(\frac{\theta_{j-1}+\theta_j}{2}\right)\frac{\phi_{j-1}-\phi_j}{h^2}\right]\right\}\\
&-R\cos(\phi_j)\sin(\theta_j)\left[\frac{F_j^X}{2}-\frac{\alpha h-2}{\alpha h+2}\left(\frac{F_j^X}{2}-\frac{m(X_{j-1}-X_j)}{h^2}\right)+\frac{m(X_j-X_{j+1})}{h^2}\right]\\
&-R\sin(\phi_j)\sin(\theta_j)\left[\frac{F_j^Y}{2}-\frac{\alpha h-2}{\alpha h+2}\left(\frac{F_j^Y}{2}-\frac{m(Y_{j-1}-Y_j)}{h^2}\right)+\frac{m(Y_j-Y_{j+1})}{h^2}\right]\\
&-I_A\sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\left[\frac{\psi_j-\psi_{j+1}}{h^2}-\sin\left(\frac{\theta_j+\theta_{j+1}}{2}\right)\frac{\phi_j-\phi_{j+1}}{h^2}\right]=0.
\end{aligned}
\tag{A.5}
$$